

EES351 2021/1 Part III.1 Dr.Prapun

6 Sampling and Reconstruction

6.1 Sampling

Definition 6.1. Sampling is the process of taking a (sufficient) number of discrete values of points on a waveform that will hopefully define the shape of the waveform.

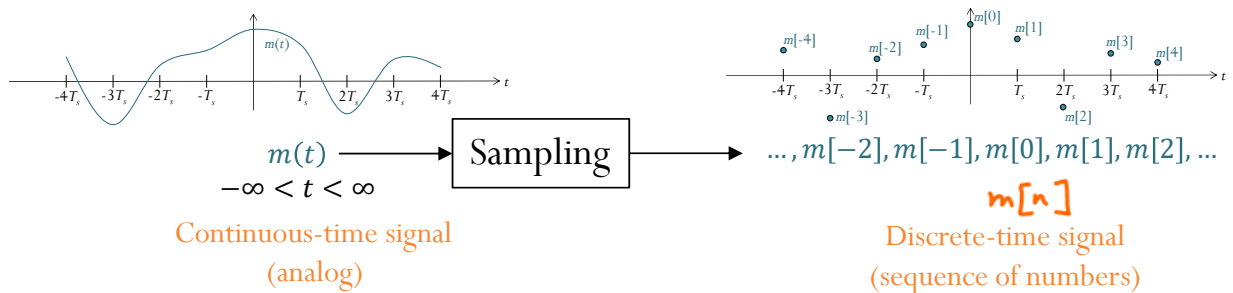


Figure 47: The Sampling Process

- In this class, the signal is sampled at a uniform rate, once every T_s seconds.

$$m[n] = m(nT_s) = m(t)|_{t=nT_s}$$

- We refer to T_s as the **sampling period**, and to its reciprocal $f_s = 1/T_s$ as the **sampling rate** which is measured in samples/sec [Sa/s].

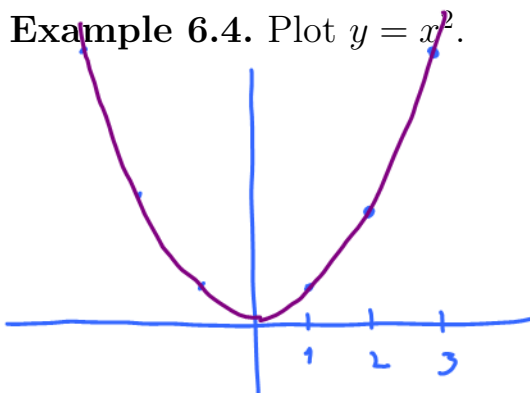
- At this stage, we assume “infinite” precision (no quantization) for each value of $m[n]$.
- The reverse process is called “reconstruction”.

6.2. Sampling = loss of information? If not, how can we recover the original waveform back.

- The more samples you take, the more accurately you can define a waveform.
- Obviously, if the sampling rate is too low, you may experience distortion (aliasing).
- The sampling theorem, to be discussed in the section, says that when the waveform is band-limited, if the sampling rate is fast enough, we can reconstruct the waveform back and hence there is no loss of information.
 - This allows us to replace a continuous time signal by a discrete sequence of numbers.
 - Processing a continuous time signal is therefore equivalent to processing a discrete sequence of numbers.
 - In the field of communication, the transmission of a continuous time message reduces to the transmission of a sequence of numbers.

Example 6.3. Mathematical functions are frequently displayed as continuous curves, even though a finite number of discrete points was used to construct the graphs. If these points, or samples, have sufficiently close spacing, a smooth curve drawn through them allows us to interpolate intermediate values to any reasonable degree of accuracy. It can therefore be said that the continuous curve is adequately described by the sample points alone.

Example 6.4. Plot $y = x^2$.



- sampling
- ① “calculate” $y = x^2$ at many values of x
 - ② “connect” the dots
- reconstruction
(interpolation)

Example 6.5. In Figure 48, we plot the function $g(t) = \sin(100\pi t)$ from 0 to 1 by connecting the values of the function at fifty uniformly-spaced points.

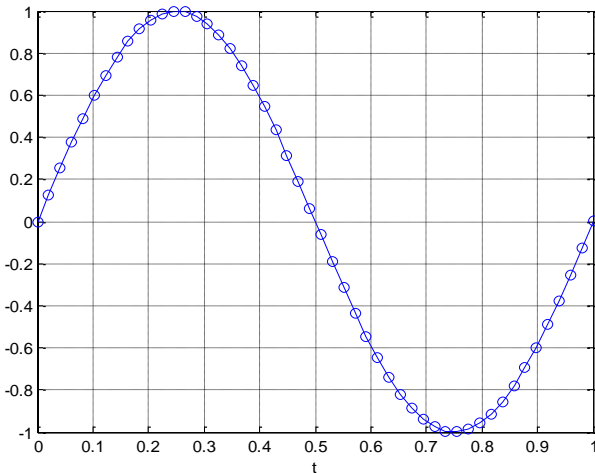


Figure 48: Plot of the function $g(t) = \sin(100\pi t)$ from 0 to 1 by connecting the values at fifty uniformly-spaced points.

Although the plot shows the correct shape of the sine wave, the **perceived frequency** is just 1 Hz.

[version 2]

Theorem 6.6. Sampling Theorem: In order to (correctly and completely) represent an analog signal, **the sampling frequency, f_s , must be at least twice the highest frequency component** of the analog signal.

6.7. If the conditions of the sampling theorem are not satisfied, we experience an effect called **aliasing** in which different signals become indistinguishable (or aliases of one another) when sampled.

- The term “aliasing” also refers to the distortion or artifact that results when the signal reconstructed from samples is different from the original continuous signal.

Example 6.8. In Example 6.5, the frequency of the sine wave is 50 Hz. Therefore, we need the sampling frequency to be at least 100.

6.9. For now, instead of trying to infer the “perceived” frequency by analyzing the plot of the function in the time domain, it is easier to use our **plotspect function to visualize** the location of the peaks (of the delta functions) in the frequency domain.

Example 6.10. Suppose the sampling frequency is 200 samples/sec. The analog signal should not have the frequency higher than 100 Hz. This is illustrated in Figure 49 in which cosine functions of different frequencies are sampled with $f_s = 200$.

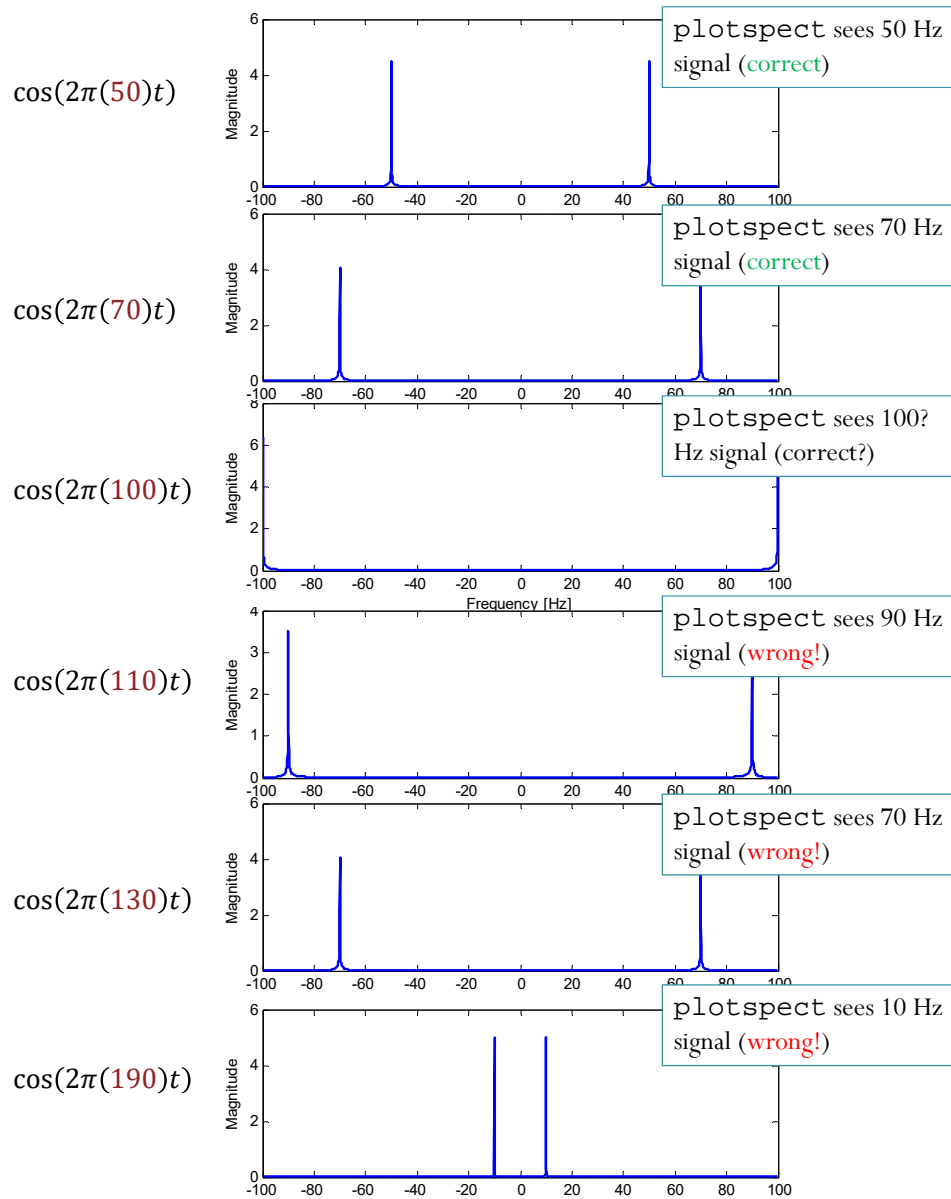


Figure 49: Our `plotspect` function can be used to visualize the “perceived” frequency of a sampled signal.

6.11. Steps to find the “perceived” frequency of the sampled signal when the sampling rate is f_s :

Assume $f_0 \geq 0$, the perceived freq. is also ≥ 0

(a) For $\cos(2\pi(f_0)t)$, we may use the "folding technique":

- (i) Consider the window of frequency from 0 to $\frac{f_s}{2}$.
- (ii) Start from 0, increase the frequency to f_0 .
Fold back at 0 and $\frac{f_s}{2}$ if necessary.

Remark: By the symmetry in the spectrum of cosine, we can always give a nonnegative answer for the perceived frequency.

(b) For $e^{j2\pi(f_0)t}$, we use the "tunneling technique":

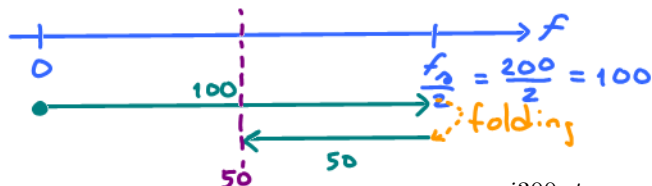
- (i) Consider the window of frequency from $-\frac{f_s}{2}$ to $\frac{f_s}{2}$.
- (ii) Start from 0.
 - i. If $f_0 > 0$, increase the frequency to f_0 (going to the right).
Restart at $-\frac{f_s}{2}$ when $\frac{f_s}{2}$ is reached.
 - ii. If $f_0 < 0$, decrease the frequency to f_0 (going to the left).
Restart at $+\frac{f_s}{2}$ when $-\frac{f_s}{2}$ is reached.

(c) We will study a more general analysis in Section 6.3.

Example 6.12. Find the perceived frequency of $\cos(300\pi t)$ when the sampling rate is 200 [Sa/s].

$$f_s = 200$$

$$f_0 = \frac{300\pi}{2\pi} = 150 \text{ Hz}$$

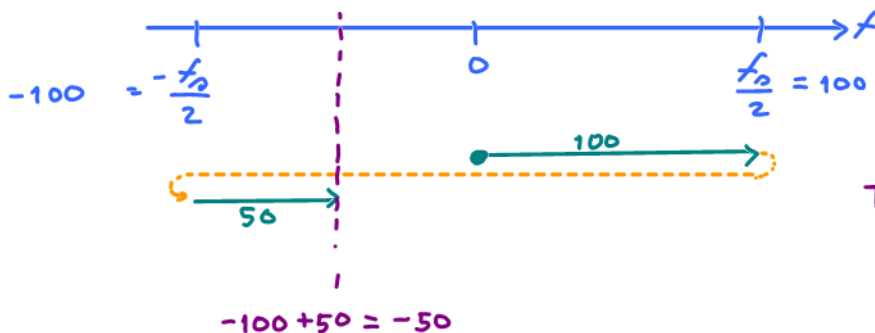


The perceived freq. = 50 Hz

Example 6.13. Find the perceived frequency of $e^{j300\pi t}$ when the sampling rate is 200 [Sa/s].

$$f_s = 200$$

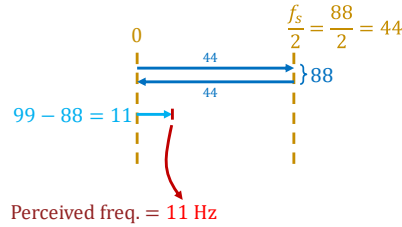
$$f_0 = 150 \text{ Hz}$$



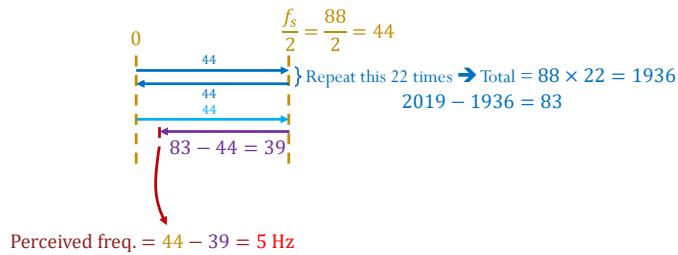
The perceived freq. = -50 Hz.

Example 6.14. In each part below, for the given value of f_0 , find the perceived frequency of $\cos(2\pi f_0 t)$ when the sampling rate is 88 [Sa/s].

a) $f_0 = 99$

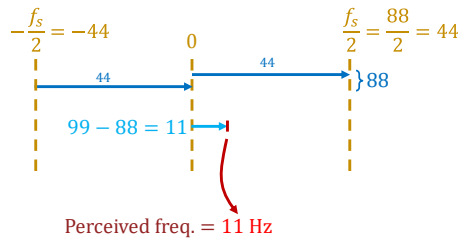


b) $f_0 = 2019$

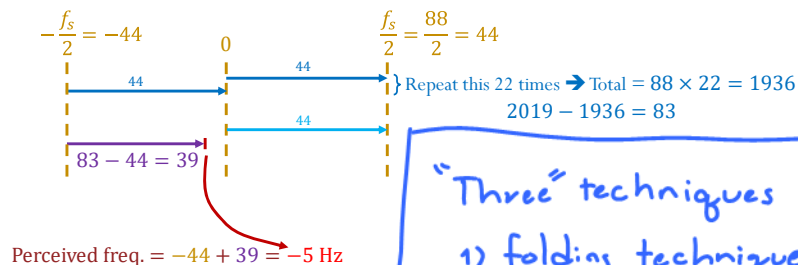


Example 6.15. In each part below, for the given value of f_0 , find the perceived frequency of $e^{j2\pi f_0 t}$ when the sampling rate is 88 [Sa/s].

a) $f_0 = 99$



b) $f_0 = 2019$



Three techniques to analyze sampling:

- 1) folding technique → analyze cos
- 2) tunneling technique → analyze complex expo.
- 3) ideal sampling → any signals
- (4) DTFT

6.16. A cosine function at frequency f_0 can also be thought of as a combination of two complex exponential at frequency f_0 and $-f_0$. Therefore, we can also use the tunneling technique to analyze the cosine function as well by looking at its individual complex-exponential components.

Example 6.17. Let's consider a signal that is closer to Example 6.5. Suppose we consider $\cos(100\pi t)$. The sampling rate used is 49 [Sa/s]. Find the perceived frequency.

① Folding technique:

② Tunneling technique

$f_0 = \frac{100\pi}{2\pi} = 50 \text{ Hz}$ $f_s = 49$

$\cos(100\pi t) = \frac{e^{j100\pi t} + e^{-j100\pi t}}{2}$
 perceived as
 $\cos(2\pi(1)t) = \frac{e^{j2\pi(1)t} + e^{j2\pi(-1)t}}{2}$

Example 6.18. Now, let's consider the signal $\sin(100\pi t)$ discussed in Example 6.5. Again, the sampling rate used is 49 [Sa/s]. Find the perceived signal.

$\sin(100\pi t) = \frac{e^{j100\pi t} - e^{-j100\pi t}}{2j}$
 perceived as
 $\sin(2\pi(1)t) = \frac{e^{j2\pi(1)t} - e^{j2\pi(-1)t}}{2j}$

(from Ex. 6.17)

Example 6.19. Application of the sampling theorem: In telephony, the usable voice frequency band ranges from approximately 300 Hz to 3400 Hz. The bandwidth allocated for a single voice-frequency transmission channel is usually 4 kHz, including guard bands, allowing a sampling rate of 8 kHz to be used as the basis of the pulse code modulation system used for the digital PSTN.

Definition 6.20.

- (a) Given a sampling frequency, f_s , the Nyquist frequency is $f_s/2$.
- (b) Given the highest (positive-)frequency component f_{\max} of an analog signal,
 - (i) the Nyquist sampling rate is $2f_{\max}$ and
 - (ii) the Nyquist sampling interval is $1/(2f_{\max})$.

